EQUAL-STRENGTH HOLE IN A PLATE IN AN INHOMOGENEOUS STRESS STATE

N. I. Ostrosablin

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Let the state of stress in a continuous plate be defined by Kolosov–Muskhelishvili functions [1]:

$$\frac{1}{2}(\sigma_x+\sigma_y)=\Phi_0(z)+\overline{\Phi_0(z)},\quad \frac{1}{2}(\sigma_y-\sigma_x)+i\tau_{xy}=\bar{z}\Phi_0'(z)+\Psi_0(z), \tag{1}$$

where $\Phi_0(z)=\varphi_0'(z), \Psi_0'(z)=\psi_0'(z)$ are known functions of the complex variable $z=x+iy$ holomorphic in the region of the plate that satisfy given conditions at the boundary of the plate.

In the plate we make a hole with edge L to which we apply constant normal and tangential stresses:

$$\sigma_n=p,\ \tau_{nt}=\tau,\ z\in L, \tag{2}$$

where (n, t) is a coordinate system linked to the normal and tangent to the contour L and oriented in the same way as the (x, y) coordinate system. On traversing L, the region occupied by the material remains on the left. There is stress redistribution on account of the hole in the plate. The stress state in the plate with the hole may be represented via the functions

$$\Phi(z)=\Phi_0(z)+\Phi_1(z),\ \Psi(z)=\Psi_0(z)+\Psi_1(z), \tag{3}$$

where $\Phi_1(z), \Psi_1(z)$ characterize the additional state of stress caused by the hole. These functions must be such that conditions (2) are met on L, while the stresses become those of (1) at the outer boundary of the plate.

We solve the problem approximately on the assumption that the dimensions of the plate are much larger than those of the hole. Then we get a problem with zero conditions at the infinitely remote edge for the additional stresses.

Apart from conditions [2] we specify further that the stress σ_t on L be constant (an equal-strength hole [2]):

$$\sigma_t=q=\text{const},\ z\in L. \tag{4}$$

The problem may not have a solution for a given L, so L is not specified in advance but is chosen such as to meet (4). Such holes may be optimal in the sense of minimal stress concentration [3-6].

The principal vector of (2) for the external forces applied to L is zero, and the functions $\Phi_0(z), \Psi_0(z)$ are holomorphic in the continuous plate, so the functions $\Phi_1(z)$, $\Psi_1(z)$ have the following order [1] near an infinitely remote point:

$$\Phi_1(z)=O(z^{-2}),\ \Psi_1(z)=O(z^{-2}). \tag{5}$$

On L we have [1]

$$^1/_2(\sigma_x+\sigma_y)=\Phi(z)+\overline{\Phi(z)}={}^1/_2(\sigma_n+\sigma_t),\quad z\in L,\ {}^1/_2(\sigma_y-\sigma_x)+i\tau_{xy}$$
$$=\bar{z}\Phi'(z)+\Psi(z)=-[{}^1/_2(\sigma_t-\sigma_n)+i\tau_{nt}]d\bar{z}/dz.$$

We substitute (2)-(4) into these equations and get the following boundary conditions for $\Phi_1(z), \Psi_1(z)$:

$$\Phi_1(z)+\overline{\Phi_1(z)}={}^1/_2(p+q)-[\Phi_0(z)+\overline{\Phi_0(z)}],\quad \bar{z}\Phi_1'(z)+\Psi_1(z)=-\alpha d\bar{z}/dz-[\bar{z}\Phi_0'(z)+\Psi_0(z)],\ z\in L, \tag{6}$$

where $\alpha={}^1/_2(q-p)+i\tau$.

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Let $\Phi_0(z)$, $\Psi_0(z)$ take the form

$$\Phi_0(z) = \sum_{k=0}^{m} a_k z^k, \quad \Psi_0(z) = \sum_{k=0}^{m} b_k z^k, \tag{7}$$

where the coefficients are assumed to be known constants. We take the functions in the form of (7) to obtain solutions of practical interest [7].

We perform conformal mapping with the functions

$$z = c\omega(\zeta) = c\zeta \sum_{n=0}^{\infty} c_n \zeta^{-n}, \quad c = |c| > 0, \quad c_0 = 1, \quad c_1 = 0 \tag{8}$$

for the region $|\zeta| > 1$ into an infinite region outside L. It can be assumed in (8) that $c = |c| > 0$, $c_1 = 0$, since one can always make the substitution

$$z = e^{i \arg c}(z_1 + |c|c_1),$$

which amounts to rotating the coordinate system through the angle arg c and then transferring the origin to the point $|c|c_1$. We put

$$\Phi_1(z) = \Phi_1[c\omega(\zeta)] = \Phi_1(\zeta), \ \Psi_1(z) = \Psi_1[c\omega(\zeta)] = \Psi_1(\zeta),$$

and get that $\Phi_1(\zeta)$, $\Psi_1(\zeta)$ in the region of an infinitely remote point have the following orders on the basis of (5) and (8):

$$\Phi_1(\zeta) = O(\zeta^{-2}), \ \Psi_1(\zeta) = O(\zeta^{-2}). \tag{9}$$

We assume that $\omega+'(t)$ satisfies Gelder's condition, and then [8]

$$\omega'+(t) = \omega+'(t), \tag{10}$$

where the plus denotes the limiting values of the function as ζ tends from the region $|\zeta| > 1$ to the points on unit circle $|t| = 1$.

We now use (7), (8), and (10) to rewrite (6) for $\Phi_1(\zeta)$, $\Psi_1(\zeta)$ (the plus sign is omitted) as

$$\Phi_1(t) + \overline{\Phi_1(t)} = \frac{1}{2}(p+q) - \left[\sum_{k=0}^{m} \alpha_k \omega^k(t) + \sum_{k=0}^{m} \overline{\alpha}_k \overline{\omega^k(t)}\right] = f(t),$$

$$\frac{\overline{\omega(t)}}{\omega'(t)} \Phi_1'(t) + \Psi_1(t) = \alpha \frac{\overline{\omega'(t)\, t^2}}{\omega'(t)} - \left[\overline{\omega(t)} \sum_{k=1}^{m} k\alpha_k \omega^{k-1}(t) + \sum_{k=0}^{m} \beta_k \omega^k(t)\right] = h(t), \quad |t| = 1, \tag{11}$$

where $\alpha_k = a_k c^k$; $\beta_k = b_k c^k$; $k = \overline{0, m}$; we find the representation of $\omega^k(\zeta)$ for large $|\zeta|$, and on the basis of (8) get

$$\omega^k(\zeta) = \zeta^k \left(\sum_{n=0}^{\infty} c_n \zeta^{-n}\right)^k = \zeta^k \sum_{n=0}^{\infty} c_n^{(k)} \zeta^{-n} = \sum_{n=0}^{k} c_{k-n}^{(k)} \zeta^n + \sum_{n=1}^{\infty} c_{k+n}^{(k)} \zeta^{-n}, \tag{12}$$

where the coefficients $c_n^{(k)}$ are determined [9] from recurrence formulas:

$$c_0^{(k)} = c_0^k = 1, \quad c_n^{(k)} = \frac{1}{nc_0} \sum_{j=1}^{n} [j(k+1) - n]\, c_j c_{n-j}^{(k)}, \quad n = \overline{1, \infty}. \tag{13}$$

We write out the combination

$$\begin{aligned} h(t)\omega'(t) - \overline{\omega(t)} f'(t) &= \alpha\overline{\omega'(t)\, t^2} - \sum_{k=1}^{m} k\overline{\alpha}_k \overline{\omega^k(t)\, \omega'(t)\, t^2} - \sum_{k=0}^{m} \beta_k \omega^k(t)\omega'(t) \\ &= \left[-\alpha\overline{\omega(t)} + \sum_{k=1}^{m} \frac{k\overline{\alpha}_k}{k+1} \overline{\omega^{k+1}(t)} - \sum_{k=0}^{m} \frac{\beta_k}{k+1} \omega^{k+1}(t)\right]' = g'(t). \end{aligned} \tag{14}$$

The boundary-value problem of (9) and (11) has been considered elsewhere [10]; we use the results of [10] with (12) and (14) to get a solution to the boundary-value problem of (11) (motion around the circle $|t|$ is clockwise):

$$\Phi_1(\zeta) = \frac{1}{2\pi i}\int \frac{f(t)}{t-\zeta}\,dt = -\left\{\sum_{k=1}^{m} \alpha_k \left[\omega^k(\zeta) - \sum_{n=0}^{k} c_{k-n}^{(k)} \zeta^n\right] + \sum_{k=1}^{m} \bar{\alpha}_k \sum_{n=1}^{k} \bar{c}_{k-n}^{(k)} \zeta^{-n}\right\}; \tag{15}$$

$$\Psi_1(\zeta)\,\omega'(\zeta) = \frac{1}{2\pi i}\int \frac{g'(t)}{t-\zeta}\,dt = \frac{d}{d\zeta}\left(\frac{1}{2\pi i}\int \frac{g(t)}{t-\zeta}\,dt\right)$$

$$= \frac{d}{d\zeta}\left\{-\frac{\alpha\bar{c}_0}{\zeta} + \sum_{k=1}^{m} \frac{k\bar{\alpha}_k}{k+1} \sum_{n=1}^{k+1} \frac{\bar{c}_{k+1-n}^{(k+1)}}{\zeta^n} - \sum_{k=0}^{m} \frac{\beta_k}{k+1}\left[\omega^{k+1}(\zeta) - \sum_{n=0}^{k+1} c_{k+1-n}^{(k+1)} \zeta^n\right]\right\}$$

$$= \frac{\alpha\bar{c}_0}{\zeta^2} - \sum_{k=1}^{m} \frac{k\bar{\alpha}_k}{k+1} \sum_{n=1}^{k+1} \frac{n\bar{c}_{k+1-n}^{(k+1)}}{\zeta^{n+1}} - \sum_{k=0}^{m} \frac{\beta_k}{k+1}\left[(k+1)\,\omega^k(\zeta)\,\omega'(\zeta) - \sum_{n=1}^{k+1} n c_{k+1-n}^{(k+1)} \zeta^{n-1}\right], \tag{16}$$

Then from the first condition of (9) we must have

$$\frac{1}{2}(p+q) - \sum_{k=0}^{m} \left(\alpha_k c_k^{(k)} + \bar{\alpha}_k \bar{c}_k^{(k)}\right) = 0; \tag{17}$$

$$\sum_{k=1}^{m} \alpha_k c_{k+1}^{(k)} + \sum_{k=1}^{m} \bar{\alpha}_k \bar{c}_{k-1}^{(k)} = 0. \tag{18}$$

We see from (16) that the second condition of (9) is obeyed. From (16) we have

$$\Psi_1(\zeta) = -\sum_{k=0}^{m} \beta_k \omega^k(\zeta) + \frac{1}{\omega'(\zeta)}\left(\frac{\alpha\bar{c}_0}{\zeta^2} - \sum_{k=1}^{m} \frac{k\bar{\alpha}_k}{k+1} \sum_{n=1}^{k+1} \frac{n\bar{c}_{k+1-n}^{(k+1)}}{\zeta^{n+1}} + \sum_{k=0}^{m} \frac{\beta_k}{k+1} \sum_{n=1}^{k+1} n c_{k+1-n}^{(k+1)} \zeta^{n-1}\right). \tag{19}$$

We determine the constant q from (17), while (17) gives that the constant α is

$$\alpha = \frac{1}{2}(q-p) + i\tau = \sum_{k=0}^{m} \left(\alpha_k c_k^{(k)} + \bar{\alpha}_k \bar{c}_k^{(k)}\right) - p + i\tau. \tag{20}$$

The function $\omega(\zeta)$ satisfies the following functional equation [10]:

$$H^-(\zeta) + \bar{\omega}(1/\zeta) F^-(\zeta) = 0, \quad |\zeta| < 1, \tag{21}$$

where the functions $H^-(\zeta)$, $F^-(\zeta)$ are defined in the region $|\zeta| < 1$ by the formulas

$$H^-(\zeta) = \frac{1}{2\pi i}\int \frac{g'(t)}{t-\zeta}\,dt = \frac{d}{d\zeta}\left(\frac{1}{2\pi i}\int \frac{g(t)}{t-\zeta}\,dt\right)$$

$$= -\frac{d}{d\zeta}\left\{-\alpha\left[\bar{\omega}\left(\frac{1}{\zeta}\right) - \frac{\bar{c}_0}{\zeta}\right] + \sum_{k=1}^{m} \frac{k\bar{\alpha}_k}{k+1}\left[\bar{\omega}^{k+1}\left(\frac{1}{\zeta}\right) - \sum_{n=1}^{k+1} \frac{\bar{c}_{k+1-n}^{(k+1)}}{\zeta^n}\right]\right.$$

$$\left. - \sum_{k=0}^{m} \frac{\beta_k}{k+1} \sum_{n=0}^{k+1} c_{k+1-n}^{(k+1)} \zeta^n\right\} = -\alpha\left[\bar{\omega}'\left(\frac{1}{\zeta}\right)\frac{1}{\zeta^2} - \frac{\bar{c}_0}{\zeta^2}\right]$$

$$+ \sum_{k=1}^{m} \frac{k\bar{\alpha}_k}{k+1}\left[(k+1)\,\bar{\omega}^k\left(\frac{1}{\zeta}\right)\bar{\omega}'\left(\frac{1}{\zeta}\right)\frac{1}{\zeta^2} - \sum_{n=1}^{k+1} \frac{n\bar{c}_{k+1-n}^{(k+1)}}{\zeta^{n+1}}\right] + \sum_{k=0}^{m} \frac{\beta_k}{k+1} \sum_{n=1}^{k+1} n c_{k+1-n}^{(k+1)} \zeta^{n-1},$$

$$F^-(\zeta) = \frac{1}{2\pi i}\int \frac{f'(t)}{t-\zeta}\,dt = \frac{d}{d\zeta}\left(\frac{1}{2\pi i}\int \frac{f(t)}{t-\zeta}\,dt\right) = -\frac{d}{d\zeta}\bar{\Phi}_1\left(\frac{1}{\zeta}\right)$$

$$= \sum_{k=1}^{m} \alpha_k \sum_{n=1}^{k} n c_{k-n}^{(k)} \zeta^{n-1} - \sum_{k=1}^{m} \bar{\alpha}_k\left[k\bar{\omega}^{k-1}\left(\frac{1}{\zeta}\right)\bar{\omega}'\left(\frac{1}{\zeta}\right)\frac{1}{\zeta^2} - \sum_{n=1}^{k} \frac{n\bar{c}_{k-n}^{(k)}}{\zeta^{n+1}}\right],$$

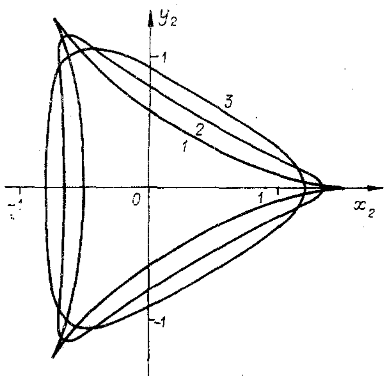

Fig. 1

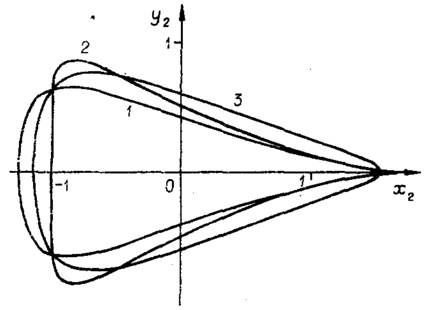

Fig. 2

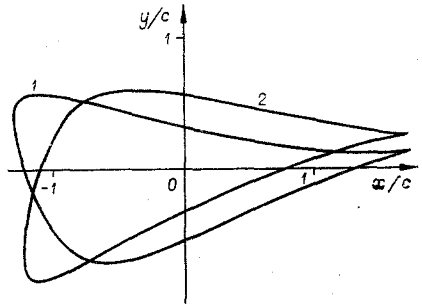

Fig. 3

where $F^-(0) = 0$, according to (18). We substitute these functions into (21) to get

$$-\alpha\left[\bar{\omega}'\left(\frac{1}{\zeta}\right)\frac{1}{\zeta^2}-\frac{\bar{c}_0}{\zeta^2}\right]+\bar{\omega}\left(\frac{1}{\zeta}\right)\left(\sum_{k=1}^{m}\alpha_k\sum_{n=1}^{k} n c_{k-n}^{(k)}\zeta^{n-1}+\sum_{k=1}^{m}\bar{\alpha}_k\sum_{n=1}^{k}\frac{n\bar{c}_{k-n}^{(k)}}{\zeta^{n+1}}\right)$$
$$+\sum_{k=0}^{m}\frac{\beta_k}{k+1}\sum_{n=1}^{k+1} n c_{k+1-n}^{(k+1)}\zeta^{n-1}-\sum_{k=1}^{m}\frac{k\bar{\alpha}_k}{k+1}\sum_{n=1}^{k+1}\frac{n\bar{c}_{k+1-n}^{(k+1)}}{\zeta^{n+1}}=0.$$

We introduce the symbols

$$A_n = n\sum_{k=n}^{m}\alpha_k c_{k-n}^{(k)}, \qquad n=\overline{1, m},$$
$$B_n = n\sum_{k=n-1}^{m}\frac{\beta_k}{k+1}c_{k+1-n}^{(k+1)}, \quad n=\overline{1, m+1}, \tag{22}$$
$$D_n = n\sum_{k=n-1}^{m}\frac{k\alpha_k}{k+1}c_{k+1-n}^{(k+1)}, \quad n=\overline{1, m+1}$$

and replace ζ by $1/\zeta$, while all the parameters are replaced by the conjugate ones, which transforms the equation to

$$-\bar{\alpha}\left[\omega'(\zeta)-c_0\right]+\sum_{n=1}^{m}\left(A_n\zeta^{n-1}+\frac{\bar{A}_n}{\zeta^{n+1}}\right)\omega(\zeta)+\sum_{n=1}^{m+1}\left(\frac{\bar{B}_n}{\zeta^{n+1}}-D_n\zeta^{n-1}\right)=0, \quad |\zeta|>1. \tag{23}$$

If $\alpha \neq 0$, we put

$$A(\zeta) = -\frac{1}{\bar{\alpha}}\sum_{n=1}^{m}\left(\frac{A_n}{n}\zeta^n-\frac{\bar{A}_n}{n}\frac{1}{\zeta^n}\right),$$
$$\gamma_0 = 0, \quad \gamma_1 = c_0-\frac{D_1}{\bar{\alpha}}, \quad \gamma_n = -\frac{D_n}{\bar{\alpha}}, \quad n=\overline{2, m+1},$$
$$\gamma_{-n} = \frac{\bar{B}_n}{\bar{\alpha}}, \quad n=\overline{1, m+1},$$

and rewrite (23) as

$$\left[\omega(\zeta)\,e^{A(\zeta)}\right]' = \sum_{n=-(m+1)}^{m+1}\gamma_n\zeta^{n-1}e^{A(\zeta)}. \tag{24}$$

The functions $e^{A(\zeta)}$, $e^{-A(\zeta)}$ can be represented as Lorant series in the region $0 < |\zeta| < \infty$:

$$e^{A(\zeta)} = \sum_{n=-\infty}^{\infty}\mu_n\zeta^n, \quad e^{-A(\zeta)} = \sum_{n=-\infty}^{\infty}\lambda_n\zeta^n,$$

where the coefficients are defined by the formulas of [11] (the motion around the circle $|\zeta| = \rho$ is counterclockwise):

$$\mu_n = \frac{1}{2\pi i} \int\limits_{|\zeta|=\rho} \frac{e^{A(\zeta)}}{\zeta^{n+1}} d\zeta, \quad 0<\rho<\infty, \quad \lambda_n = \mu_n(-\bar{\alpha}), \quad n = \overline{-\infty, \infty}.$$

Then the general solution to (24) is written as

$$\omega(\zeta) = \left[\left(\sum_{k=-(m+1)}^{m+1} \gamma_k \mu_{-k}\right) \ln \zeta + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\left(\sum_{k=-(m+1)}^{m+1} \gamma_k \mu_{n-k}\right) \zeta^n + \delta\right] e^{-A(\zeta)}, \tag{25}$$

where δ is a constant of integration.

It follows from (25) that $\omega(\zeta)$ has the form of (8) if we have the following:

$$\begin{gathered}
\sum_{k=-(m+1)}^{m+1} \gamma_k \mu_{-k} = 0, \quad \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{s}\left(\sum_{k=-(m+1)}^{m+1} \gamma_k \mu_{s-k}\right) \lambda_{-s} + \delta\lambda_0 = 0, \\
\sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{s}\left(\sum_{k=-(m+1)}^{m+1} \gamma_k \mu_{s-k}\right) \lambda_{1-s} + \delta\lambda_1 = 1, \\
\sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{s}\left(\sum_{k=-(m+1)}^{m+1} \gamma_k \mu_{s-k}\right) \lambda_{n-s} + \delta\lambda_n = 0, \quad n = \overline{2, \infty},
\end{gathered} \tag{26}$$

where the coefficients of the $\omega(\zeta)$ of (8) are

$$c_{n+1} = \sum_{\substack{s=-\infty \\ s\neq 0}}^{\infty} \frac{1}{s}\left(\sum_{k=-(m+1)}^{m+1} \gamma_k \mu_{s-k}\right) \lambda_{-n-s} + \delta\lambda_{-n}, \quad n = \overline{1, \infty}. \tag{27}$$

Therefore, the function $\omega(\zeta)$ will have the form of (8) if the parameters of the problem are such that (26) is obeyed. As $\omega(\zeta)$ satisfies (21) [10], we have that condition (18) is obeyed. Then the coefficients c_{n+1}, $n=\overline{1,\infty}$ are defined by (27). As for $m\geq 2$ the right side of (27) contains the coefficients $c_2, \ldots, c_m$, the first $m-1$ relations of (27) will be a system of equations for the coefficients $c_2, \ldots, c_m$.

If we substitute (8) into (23) and compare coefficients for identical powers of ζ, we get the following system of equations for the c_n:

$$\sum_{j=0}^{m+1-n} c_j A_{j+n-1} - D_n = 0, \quad n = \overline{1, m+1}; \tag{28}$$

$$\begin{gathered}
\sum_{j=1}^{m} A_j c_{j+1} + \bar{A}_1 c_0 = 0, \\
\sum_{j=1}^{m} A_j c_{n+1+j} + \bar{\alpha} n c_{n+1} + \sum_{j=1}^{n+1\leqslant m} \bar{A}_j c_{n+1-j} + \bar{B}_n = 0, \; n = \overline{1, m+1}, \\
\sum_{j=1}^{m} A_j c_{n+1+j} + \bar{\alpha} n c_{n+1} + \sum_{j=1}^{m} \bar{A}_j c_{n+1-j} = 0, \; n = \overline{m+2, \infty}.
\end{gathered} \tag{29}$$

The form of (21) implies, which can be checked directly, that equations (28) are obeyed identically. The first equation in (29) coincides with (18), as can be checked by using (13) and (22). Therefore, the coefficients in the mapping function $\omega(\zeta)$ should satisfy (29), where α, A_j, B_j are defined by (20) and (22).

A check shows that the c_{n+1} defined by (27) satisfy (29) if (26) is obeyed up to $n=m$, whereupon the other relations in (26) will be obeyed. Therefore, if there is to be an equal-strength hole for $\alpha\neq 0$ we must specify that the first $m+2$ relations in (26) are obeyed. One of these relations defines the constant δ.

If

$$\alpha = \sum_{k=0}^{m} (\alpha_k c_k^{(k)} + \bar{\alpha}_k \bar{c}_k^{(k)}) - p + i\tau = 0, \tag{30}$$

then $\omega(\zeta)$ is found from (23). The coefficients in the series expansion may be found from (29), with the initial coefficients $c_0, \ldots$, cm related by (30).

Therefore, the boundary-value problem of (9) and (11) has been solved: the functions $\Phi_1(\zeta)$, $\Psi_1(\zeta)$ are expressed by (15) and (19), while the coefficients in $\omega(\zeta)$ are defined by (27) when the m+2 first conditions in (26) are obeyed or else by Eqs. (29) and (30). For the solution to be complete we need to impose constants on the c_n such that the $\omega(\zeta)$ of (8) will be of one sheet in the region of $|\zeta|>1$. It can be shown that for this to be so in the region $|\zeta|\geq\rho>1$ it is sufficient to meet the condition

$$\sum_{n=1}^{\infty}\left|\frac{c_{n+1}}{c_0}\right|\frac{n}{\rho^{n+1}}<1.$$

We give some particular cases. For m=1 we have from (13), (20), and (22) that

$$A_1=\alpha_1,\ B_1=\beta_0,\ B_2=\beta_1,\ D_1=0,\ D_2=\alpha_1,$$
$$\alpha=\alpha_0+\overline{\alpha}_0-p+i\tau,\ \gamma_0=0,\ \gamma_1=1,$$
$$\gamma_2=-\alpha_1/\overline{\alpha},\ \gamma_{-1}=\overline{\beta}_0/\overline{\alpha},\ \gamma_{-2}=\overline{\beta}_1/\overline{\alpha}.$$

The values of μ_n,λ_n are

$$\mu_n=\left(\frac{\alpha_1}{|\alpha_1|}\right)^n J_n\left(-\frac{2|\alpha_1|}{\overline{\alpha}}\right),\quad \lambda_n=\left(\frac{\alpha_1}{|\alpha_1|}\right)^n J_n\left(\frac{2|\alpha_1|}{\overline{\alpha}}\right),$$

where $J_n(\ )$ are Bessel functions of order n [11]. The coefficients c_{n+1} are given by

$$c_{n+1}=\sum_{\substack{s=-\infty\\ s\neq 0}}^{\infty}\frac{1}{s}\left(\sum_{k=-2}^{2}\gamma_k\mu_{s-k}\right)\lambda_{-n-s}+\delta\lambda_{-n},\quad n=\overline{1,\infty},\tag{31}$$

where the given parameters and constant δ are such that

$$\sum_{k=-2}^{2}\gamma_k\mu_{-k}=0,\ \sum_{\substack{s=-\infty\\ s\neq 0}}^{\infty}\frac{1}{s}\left(\sum_{k=-2}^{2}\gamma_k\mu_{s-k}\right)\lambda_{-s}+\delta\lambda_0=0,$$
$$\sum_{\substack{s=-\infty\\ s\neq 0}}^{\infty}\frac{1}{s}\left(\sum_{k=-2}^{2}\gamma_k\mu_{s-k}\right)\lambda_{1-s}+\delta\lambda_1=1\tag{32}$$

is obeyed. If $\alpha=0$, then $\omega(\zeta)=(\alpha_1\zeta-\overline{\beta}_0/\zeta^2-\overline{\beta}_1/\zeta^3)\ (\alpha_1+\overline{\alpha}_1/\zeta^2)^{-1}$; if $\alpha_1=0$, then (32) is obeyed, with $\delta=0$, and from (31) we get

$$c_2=-\overline{\beta}_0/\overline{\alpha},\ c_3=-\overline{\beta}_1/2\alpha,\ \ c_{n+1}=0,\ n=\overline{3,\infty},$$

i.e.,

$$\omega(\zeta)=\zeta-\overline{\beta}_0/\overline{\alpha}\zeta-\overline{\beta}_1/2\overline{\alpha}\zeta^2.\tag{33}$$

This function is of one sheet in the region $|\zeta|>1$, if the parameters are such that

$$|\overline{b}_0|+|\overline{b}_1|c\leqslant|a_0+\overline{a}_0-p-i\tau|.\tag{34}$$

The above solutions were not obtained in [2], where the case m=1 was considered. The solution of [2] subject to condition (18) follows from (31) or from (29) if we put $c_{n+1}=0$, $n=\overline{2,\infty}$, where we get

$$c_2=-\overline{a}_1/a_1=-\overline{b}_0/\overline{\alpha}=-\overline{b}_1/\overline{a}_1.$$

This shows that an equal-strength hole in that case is a degenerate ellipse, not simply an ellipse [2, 12], and this hole does not alter the initial state of stress, since the functions $\Phi_1(\zeta)$, $\Psi_1(\zeta)$ will be zero.

Figures 1-3 show the equal-strength holes corresponding to (33) for certain values of the parameters. We write the coefficients c_2 and c_3 in the exponential form

$$c_2=|c_2|e^{i\varphi_2},\quad c_3=|c_3|e^{i\varphi_3}.$$

Figure 1 shows the L for the following values of the parameters: $c_2=0$, $|c_3|=1/2;\ 1/3;\ 1/5$ for curves 1-3 respectively, where $z_2=(z/c)e^{-i\varphi_3/3}$. For $c_2=0$, the equal-strength holes are hypotrochoids. The parameters in Fig. 2 are as follows: $\varphi_2/2=\varphi_3/3$, $|c_2|=2|c_3|=1/2$ (curve 1), $|c_2|=|c_3|=1/3$ (curve 2), and $|c_2|=1/3$, $|c_3|=1/5$ (curve 3), where $z_2=(z/c)e^{-i\varphi_2/2}$. The parameters in Fig. 3 are $|c_2|=2|c_3|=1/2$, $\varphi_2=0$, $\varphi_3=\pi/6$ (curve 1), $\varphi_2=\pi/6$, $\varphi_3=0$ (curve 2). The holes with nodal points correspond to parameters for which the sign of equality applies in (34).

Equations (26) are obeyed identically if $\alpha_k = 0$, $k = \overline{1, m}$, and $\delta = 0$; then (27) becomes as follows on the basis of (22) for the B_n:

$$c_{n+1} = -\frac{1}{\alpha} \sum_{k=n-1}^{m} \frac{\bar{\beta}_k}{k+1} \bar{c}_{k+1-n}^{(k+1)}, \quad n = \overline{1, m+1}, \ c_{n+1} = 0, \ n = \overline{m+2, \infty}. \tag{35}$$

For $n = m$, $m+1$, we have from (35) with (13) that

$$c_{m+1} = -\bar{\beta}_{m-1}/\bar{\alpha} m, \quad c_{m+2} = -\bar{\beta}_m/\bar{\alpha}(m+1).$$

For $m \geq 2$ the other expressions in (35) will constitute a system of equations for the coefficients $c_2, \ldots, c_m$; for example, for $m = 3$ the equations are

$$c_2 = -\frac{1}{\alpha}(\bar{\beta}_0 + \bar{\beta}_2 \bar{c}_2 + \bar{\beta}_3 \bar{c}_3), \ c_3 = -\frac{1}{\alpha}\left(\frac{\bar{\beta}_1}{2} + \bar{\beta}_3 \bar{c}_2\right),$$

from which we get

$$c_2 = \frac{-\left(\bar{\delta}_0 - \frac{1}{2}\delta_1\bar{\delta}_3\right)(1 - \delta_3\bar{\delta}_3) + \left(\delta_0 - \frac{1}{2}\bar{\delta}_1\delta_3\right)\bar{\delta}_2}{(1 - \delta_3\bar{\delta}_3)^2 - \delta_2\bar{\delta}_2}, \quad c_3 = -\frac{\bar{\delta}_1}{2} - \bar{\delta}_3\bar{c}_2,$$

where $\delta_k = \beta_k/\alpha$; $k = \overline{0.3}$. For $m > 3$ the system of (35) becomes nonlinear and will have several solutions, i.e., there may be several equal-strength holes for the given parameters. It may be that the specificiations at $\omega(\zeta)$ is on one sheet and will rule out some of the solutions.

If $\alpha_k = 0$, $k = \overline{1, m}$, we see from (15) that $\Phi_1(\zeta)$ is zero, i.e., the hole in that case is not only of equal strength but is also harmonic [13].

Therefore, a distinction of the present study from [13] is that we have defined equal-strength holes in a plate with an inhomogeneous initial stress distribution. These holes can also be harmonic.

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